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THE PROPAGATION OF A WEAK DISCONTINUITY IN A NON-LINEAR MEDIUM WITH RIGID UNLOADING*

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The unloading wave which arises in a non-linear semi-infinite rod when a smooth load is applied to its end is investigated. The unloading of the material of the rod is assumed to be rigid. A similar problem has been considered in /1/ (also, see /2/, p.104) where the stress at the end of the rod was specified. The solution of this problem was reduced in /1/ to an implicit functional relationship which was subsequently analysed using asymptotic methods. The formulation in this paper differs from that in /1/ only in that it is not the stress but the velocity which is specified at the end of the rod. However, with such a formulation, the solution of the problem can be successfully obtained in explicit form. The connection between the results obtained in this paper and the results in /1/ is pointed out.

1. Let us consider a semi-infinite rod with a density $\rho = \mathrm{const}$ which is arranged on the semi-axis $x \geqslant 0$. We assume that, under active loading, the stresses and deformations in the rod are connected by the relationship

$$\sigma = b(\varepsilon),\ b' > 0,\ b'' \geqslant 0 \tag{1.1}$$

while the deformations remain constant for unloading. Next, we assume that, at the instant of time $t = 0$, a load is applied to the end of the rod which is in an unstressed, resting state. We specify this load in the following manner:

$$v(t, 0) = v_0(t) \geqslant 0 \tag{1.2}$$

(v is the velocity of a material element). In doing this, we assume that the function $v_0(t)$ differs from zero only in the interval $(0, T)$, it increases monotonically in the interval $(0, t_m)$ and monotonically decays in the interval (t_m, T). Hence, the function $v_0(t)$ reaches a unique maximum when $t = t_m$.

The problem involves the determination of the equation for the unloading wave front. This equation is sought in the form $x = \varphi(t)$. We shall denote by t^* the previously unknown point on the t axis from which the unloading wave front starts out. Hence, $\varphi(t^*) = 0$.

2. In the loading domain lying in the t, x plane ahead of the unloading front the wave motion will be described by a simple rarefaction wave (shock waves are therefore not formed). In fact, the equation of a simple wave is valid in Lagrangian coordinates in the case of deformations and the solution of this equation, as is well-known, can be written in the form

$$\varepsilon = \varepsilon_0\left(t - x/\sqrt{b'(\varepsilon)/\rho}\right) \tag{2.1}$$

where ε_0 is a certain function. Furthermore, one of the Riemann invariants is identically zero in the loading domain:

$$v + g(\varepsilon) = 0;\quad g(\varepsilon) \equiv \int_0^{\varepsilon} \sqrt{\frac{b'(\varepsilon)}{\rho}}\, d\varepsilon \tag{2.2}$$

Consequently,

$$v_0(t) = -g(\varepsilon_0(t)) \tag{2.3}$$

By now applying the function g to both sides (2.1), we obtain the formula for the velocities in the active loading domain using (2.2) and (2.3)

$$v = v_0\left(t - x/\sqrt{b' \circ g^{-1}(-v)/\rho}\right) \tag{2.4}$$

(the small circle denotes the superpositioning of functions).

3. Let us now write (in Lagrangian coordinates) the equations of motion in the domain of rigid unloading

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$$\partial\sigma/\partial x = \rho\partial v/\partial t, \ \partial v/\partial x = 0 \tag{3.1}$$

whereupon we immediately obtain

$$v = v_0(t), \ t > t^* \tag{3.2}$$

We recall that the quantity t^* is as yet unknown.

Let us now utilize the condition for the continuity of the velocities on the unloading wave front. By substituting $\varphi(t)$ instead of x and $v_0(t)$ instead of v into the right-hand side of inequality (2.4), we have from (2.4) and (3.2)

$$v_0(t) = v_0(t - \varphi(t)/\sqrt{b'\circ g^{-1}(-v_0(t))/\rho}) \tag{3.3}$$

This functional equation is remarkable in that the arguments are under the sign of one and the same function on the right- and the left-hand sides. This fact enables us to solve this equation in an explicit form.

Actually, for each $t > t^*$ we denote the value of the time by $\bar{t}$ such that $v_0(t) = v_0(\bar{t}), \bar{t} < t$. Then, since, according to the condition, the function $v_0(t)$ consists only of two segments where it is monotonic, from the functional Eq.(3.3) we get the required equation for the unloading wave front (the argument on the right-hand side of (3.3) under the sign of v_0 is equated to $\bar{t}$):

$$\varphi(t) = (t - \bar{t})/\sqrt{b'\circ g^{-1}(-v_0(t))/\rho} \tag{3.4}$$

We finally recall that $\varphi(t^*) = 0$. However, it follows from (3.4) that this equality is only possible if $\bar{t}^* = t^*$. Consequently, t^* is the point of the maximum on the curve $v_0(t)$, that is, $t^* = t_m$.

4. Let us now show that the stress field which arises in the rigid unloading domain can be recovered such that the condition of the continuity of the stresses on the unloading wave front is satisfied.

We have from (3.1) and (3.2) in the unloading domain that

$$\partial\sigma/\partial x = \rho v_0'(t), \ t > t_m$$

whence

$$\sigma(t, x) = \rho v_0'(t)\, x + F(t) \tag{4.1}$$

where $F(t)$ is a certain function to be defined. In particular, at the front we have

$$\sigma(t, \varphi(t)) = \rho v_0'(t)\, \varphi(t) + F(t) \tag{4.2}$$

On the one hand, in the region of active stress (where a simple wave propagates), equality (2.2) and the defining relationship (1.1) yield

$$\sigma = b\circ g^{-1}(-v) \tag{4.3}$$

whence

$$\sigma(t, \varphi(t)) = b\circ g^{-1}(-v(t, \varphi(t)) \equiv b\circ g^{-1}(-v_0(t)), \ t > t_m \tag{4.4}$$

By equating the right-hand sides of (4.2) and (4.4) and taking account of expression (3.4) for $\varphi(t)$, we finally obtain

$$F(t) = b\circ g^{-1}(-v_0(t)) - v_0'(t)(t - \bar{t})\sqrt{\rho b_0'\circ g^{-1}(-v_0(t))} \tag{4.5}$$

Thus, formulae (4.1) and (4.5) explicitly recover the stress field in the unloading region which satisfies the continuity condition at the front $x = \varphi(t)$.

5. Let us use the notation $\sigma_0(t) \equiv \sigma(t, 0)$. Then it follows from (4.1) and (4.5) that

$$\sigma_0(t) = b\circ g^{-1}(-v_0(t)) - v_0'(t)(t - \bar{t}) \times \sqrt{\rho b'\circ g^{-1}(-v_0(t))}, \ t > t_m \tag{5.1}$$

while, by virtue of (4.3),

$$\sigma_0(t) = b\circ g^{-1}(-v_0(t)), \ t < t_m \tag{5.2}$$

It can be seen from the last two formulae that, in the case of a function $v_0(t)$ which satisfies the conditions formulated in Sect.1, the function $\sigma_0(t)$ turns out to be a non-negative monotonically increasing function in $(0, t_m)$ and a monotonically decaying function in (t_m, T). The result which has been obtained is in accord with the conclusion that the unloading front emerges from the point t_m of the t axis.

Remark. Formula (4.4) yields an expression for the mangitude of the stress at the front $\sigma = \sigma(t, \varphi(t))$ in terms of the boundary value of the velocity $v_0(t)$. By solving Eq.(4.4) for $v_0(t)$ and eliminating $v_0(t)$ from (5.1), it is possible to obtain an ordinary differential equation which relates $\sigma = \sigma(t, \varphi(t))$ and $\sigma_0(t)$:

$$d\sigma/dt = (\sigma_0(t) - \sigma)/(t - \bar{t}) \tag{5.3}$$

In this formula we must assume that $\bar{t} = \bar{t}(\sigma)$, since the stress in the simple wave is constant on the characteristic joining the points $(\bar{t}, 0)$ and $(t, \varphi(t))$. (In fact, when $t > t_m$, we have, according to the construction of the point $\bar{t}$, that $v(t, \varphi(t)) = v_0(\bar{t})$, but it follows from this that $\sigma(t, \varphi(t)) = \sigma_0(t)$, whence the assertion made above also follows). A direct derivation of Eq.(5.3) has been given in /1/ where it was also shown that this equation is integrated in quadratures and a functional relationship is obtained which relates $\sigma = \sigma(t, \varphi(t))$, $\sigma_0(t)$ and t.

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THE RELATIONSHIP BETWEEN THE ENDOCHRONIC THEORY OF PLASTICITY AND THE "NEW" MEASURE OF INTERNAL TIME*

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The transition from an earlier version of the endochronic theory of plasticity (ETP) to a theory with a "new" measure of internal time is considered together with the mutual relationship between the latter and flow theory.

The endochronic theory of plasticity was initially put forward as a theory in which there was no yield surface (YS) /1/. This was its essential difference both from the well-known classical theories of plasticity (of the flow-theory type) and from the many modern theories which are based on the concept of a yield surface. Recently, however, a version of the isochronic theory of plasticity has become widely used which is based on a "new" measure of the internal time for which the Odqvist parameter /2-4/ is actually used. There is already a yield surface in this version of the theory which may be considered as a rejection by the authors of this approach of the initial idea of constructing an analytical (non-singular) plasticity functional for arbitrary complex deformation processes.

1. We shall use a vector representation of the loading and deformation processes. Let $\boldsymbol{\sigma}$ and $\mathbf{e}$ be the stress and deformation vectors respectively /5/.

The ETP functional is written in the form

$$\boldsymbol{\sigma} = \int_0^z J(z-\eta)\, d\mathbf{e}(\eta) \tag{1.1}$$

and is formally analogous to the linear viscoelasticity functional only, instead of the physical time, a new parameter z, referred to as the internal time /1/, is used to describe the history of the deformation and loading processes. Generally speaking, the internal time z is assumed to be a functional of the deformation process. Several possible definitions of this quantity have been proposed. It was initially thought that

$$dz = ds/f(s), \quad ds = |d\mathbf{e}| \tag{1.2}$$

where the function $f\ (f > 0)$ is responsible for the effects of isotropic strengthening (or weakening) of the material and is usually called the strengthening function.

Refinement of the initial version of ETP proceeded in several directions. For instance, an alternative approach to the construction of certain ETP relationships was proposed in /6/. This approach was based on a more complex tensor-parametric form of writing the plasticity functional.

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